The optimal level of government debt and wealth inequality

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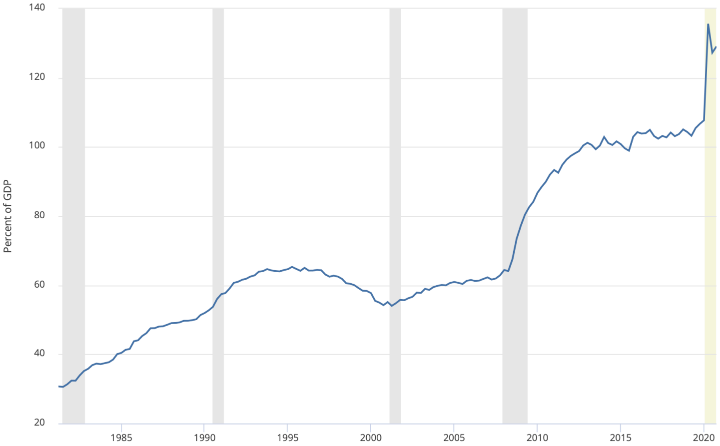
Motivation [1]: the growing government debt to GDP

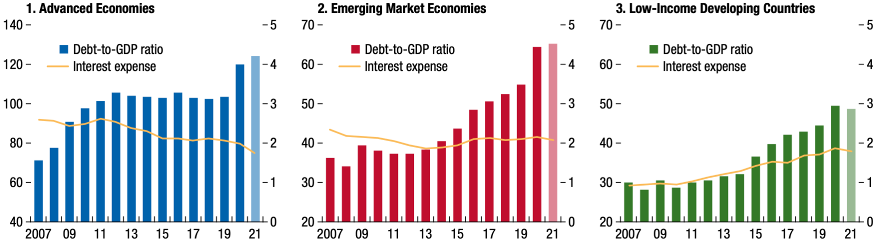
US government debt/GDP



Source: FRED https://fred.stlouisfed.org/series/GFDEGDQ188S#0. Grey areas indicate recessions in the U.S., the most recent one is ongoing.

Motivation [2]: despite rising government debt levels, interest expenses are decreasing in advanced economies

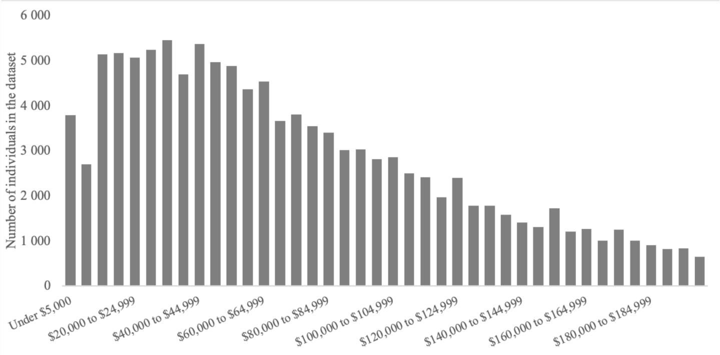
Interest expenses and government debt, 2007–2021



Source: IMF Fiscal monitor, April 2021

Motivation [3]: fat right tail of income distribution

Distribution of annual household income in the U.S. in 2019



The graph is truncated for incomes above \$200 000, the fraction of such individuals is 10,25% in the dataset.
Source: built on the data from the U.S. Census bureau
https://www.census.gov/data/tables/time-series/demo/income-poverty/cps-hinc/hinc-01.2019.html

Motivation [4]: highly non-normal distributions of income and wealth with fat right tails

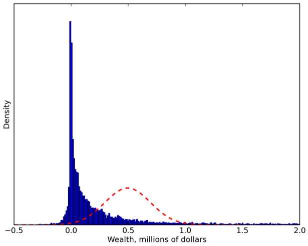


Figure: 2010 U.S. family wealth distribution and dotted normal curve

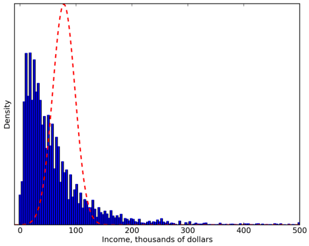


Figure: 2010 U.S. family income distribution and dotted normal curve

Source: Boik (2014), pp. 55 and 57.

Plan of the presentation

Literature review

The model

Numerical approach

Results

Optimal government debt level in the literature

Paper	Government debt/GDP level
Aiyagari and McGrattan (1998)	60%
Floden (2001)	–100%
Röhrs and Winter (2011)	–50% / –110%
Röhrs and Winter (2017)	80%
Vogel (2014)	–180% to –110%
Acikgoz (2015)	300%
Desbonnet et al. (2016)	5%
Chatterjee et al. (2017)	105%
Le Grand et al. (2017)	33%
Acikgoz (2018)	398% to 1126%
Dyrda and Pedroni (2018)	–300%

Literature review: optimal government debt

- *Aiyagari, McGrattan (1998)*: heterogeneous agents model without an aggregate shock
- *Desbonnet et al. (2016)*: idiosyncratic and aggregate shocks, 2 types of agents
- *Chatterjee et al. (2017)*: the role of government investment in public infrastructure, transitional dynamics
- *Vogel (2014)*: OLG model, welfare gains for poor, middle-class, rich agents
- *Bornstein (2020)*: continuous time, strategic default, optimal bond maturity, endogenous borrowing limit
- *Fernández-Villaverde et al. (2019)*: continuous time, financial sector, endogenous aggregate risk, machine learning techniques

Literature review: methods

Heterogenous agents models in discrete time:

- without aggregate shock: *Bewley (1983), Huggett (1993), Aiyagari (1994)*
- with aggregate shock: *Krusell and Smith (1998), Miao (2006), Boppart et al. (2017), Le Grand and Ragot (2017)*

Heterogenous agents models in continuous time:

- without aggregate shock: *Achdou et al. (2017), Ruttscheidt (2018), Rocheteau et al. (2015), Parra-Alvarez, et al. (2017)*
- with aggregate shock: *Okahata (2019), Ahn et al. (2017), Fernández-Villaverde et al. (2019)*

Literature review: income and wealth inequality

- *Hubar, Koulovatianos, Li (2020)*: portfolio choice, explanation of the risk-taking pattern by labor-income risk
- *Benhabib et al. (2018, 2019)*: skewed earnings, differential saving rates across wealth levels, idiosyncratic returns to wealth
- *Gabaix (2009)*: broad range of topics connected to the Pareto law
- *Civale et al. (2017), Tauchen (1986)*: procedure of calibration of transition probabilities
- *Guvenen (2015, 2019)*: nonparametric methods to estimate not lognormal income process
- *Kaplan (2012)*: life-cycle model, approximating the persistent component of the wage process through an 11-state Markov chain

Idea of the optimal government debt

Effects of increasing government debt on the economy:

- **Positive effect:**
  Increase in government debt $\Rightarrow$ interest rate rises, capital is crowded out $\Rightarrow$ it is less costly to hold the debt (lower opportunity cost), the borrowing constraints are relaxed, providing an opportunity for agents to smooth their consumption by holding a risk-free debt $\Rightarrow$ redistributive and insurance effect of proportional taxation and transfers, reduction in households consumption variability and wealth inequality $\Rightarrow$ increase in welfare
- **Negative effect:**
  Increase in government debt $\Rightarrow$ distortionary effect (debt is financed by marginal taxes); capital is crowded out, output and wages decrease $\Rightarrow$ reduction in consumption $\Rightarrow$ decrease in welfare

Assumptions [1]

- Based on Aiyagari and McGrattan (1998) model
- Mean-Field-Game approach to address heterogeneity from Achdou et al. (2014, 2017, 2020) papers
- closed economy
- exogenous technical progress
- large number of infinitely lived agents (due to the borrowing constraints the behavior of agents is similar to the OLG model)
- no perfect insurance market (borrowing constraints + incomplete markets) $\Rightarrow$ uninsurable idiosyncratic labor-income shock $\Rightarrow$ agents make precautionary savings or borrow

Assumptions [2]

- proportional tax on wage and capital returns to finance fiscal policy, which distorts saving decisions.
- analysis of the stationary wealth distribution and welfare maximization in the steady state
- government debt of the U.S. is considered as a risk-free asset.

Table: U.S. sovereign ratings by the main agencies

Moody's		S&P		Fitch	
date	rating	date	rating	date	rating
25.04.2018	Aaa (Stable)	10.06.2013	AA+ (Stable)	02.04.2019	AAA
02.08.2011	Aaa (Negative)	05.08.2011	AA+ (Negative)	21.03.2014	AAA (Stable)
13.07.2011	Aaa (Under Review)			21.12.2011	AAA (Negative)
15.11.2003	Aaa (Stable)			21.09.2000	AAA (Stable)
05.02.1949	Aaa			01.04.1996	AAA
				13.11.1995	AAA (Negative)
				10.08.1994	AAA

Overview

Research question: What is the optimal level of the US public debt and how the fiscal policy affects wealth inequality?

- heterogeneous agents model with government debt in continuous time: Mean-field-game approach (HJB, KF equations, finite difference approach)
- borrowing constraints, idiosyncratic income shocks ⇒ precautionary savings
- the model generates an endogenous wealth distribution, enables meaningful policy experiments
- NMAR (mixture of normals AR) process of the logarithm of labor endowments
- new: introducing 11 states, portfolio choice, rare jump shock

Answer: 120-140% of GDP

Choice of the approach

- *Why the U.S.?* - Availability of qualitative data, opportunity to compare results with other papers
- *Why not a representative agent approach?* - Inconsistent with empirical evidence, wealth inequality, differentiated effect of fiscal policy, trade of assets between agents, different MPCs, portfolio choices, return on investments
- *Why not a discrete time?* In continuous time:
  - eazy to introduce borrowing constraints
  - system of just 2 types of partial differential equations
  - more accuracy and higher speed of computation

Model with 11 states Markov chain [1]

Consumers:

$$\max_{(c_t, a_t)_{t \geq 0}} E_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

$$s.t. \quad da_t = [(1-\tau) r a_t + (1-\tau) y_{jt} - c_t + Tr] dt$$

$$a_t \geq \underline{a}$$

Model with 11 states Markov chain [2]

$y_{j,t} \in \{y_{1,t}, y_{2,t}, \cdots, y_{11,t}\} = \{w \cdot z_{1,t}, w \cdot z_{2,t}, \cdots, w \cdot z_{11,t}\}$,

Poisson process: $Pr(z_{t+dt} = z_j | z_t = z_i) = \lambda_{i,j} dt$,

Stationary labor: $L = \sum_{i=1}^{11} g_{z_i} z_i$.

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Firms:
Production function: $Y_t = TFP \cdot K_t^{\alpha} L^{1-\alpha}$

$$r_t = TFP \cdot \alpha \left(\frac{K_t}{L}\right)^{\alpha-1} - \delta, \qquad w_t = TFP \cdot (1-\alpha) \left(\frac{K_t}{L}\right)^{\alpha}$$

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Government:

$$dB_t = [(1-\tau) r B_t + G_t + Tr_t - \tau(Y_t - \delta K_t)] dt, \quad G_t = \bar{g} Y_t$$

Model with 11 states Markov chain [3]

Equilibrium in the model

- the asset market clears under the equilibrium interest rate r:

$$B + K = A = \int_{\underline{a}}^{\infty} a g_1(a, y_1) da + ... + \int_{\underline{a}}^{\infty} a g_{11}(a, y_{11}) da = S(r, y)$$

- the goods market clears: $C + \delta K + G = F(K, L)$.
- stationary distribution of assets:

$$\int_{\underline{a}}^{\infty} g_1(a, y_1) da + ... + \int_{\underline{a}}^{\infty} g_{11}(a, y_{11}) da = 1$$

where $g_j(a, y_j) \geq 0$ is a joint distribution of income and wealth.

Solution of the model

Mean-field-game approach:

- Hamilton-Jacobi-Bellman (HJB) equation
- Kolmogorov Forward (Fokker-Plank) equation

Discrete time Bellman equation:

$$V(a_t) = \max_{c_t \geq 0} \left\{ u(c_t) + \frac{1}{1+\rho} E_t[V(a_{t+1})] \right\}$$

Continuous time HJB derivation

HJB equation in the model:

$$\rho V_j(a) = \max_{c \geq 0} \big\{ u(c) + V_j'(a)((1-\tau)ra + (1-\tau)y_j - c + Tr))] +$$

$$+ \textstyle\sum_{m=1}^{11} \lambda_{j,m}[V_m(a) - V_j(a)] \Big\} \quad \forall j = \{1, ...11\}$$

FOCs

For CRRA utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

$$u'(c) = V'(a) \Rightarrow c^{-\gamma} = V'(a) \Rightarrow c = [V'(a)]^{-1/\gamma}$$

$$s_j = (1-\tau)ra + (1-\tau)y_j - c + Tr$$

Boundary condition:

$$V_j'(\underline{a}) \geq u'((1-\tau)r\underline{a} + (1-\tau)y_j + Tr)$$

Kolmogorov forward (Fokker-Planck) equation

For general Itô process

$$dz_t = \mu(z_t)dt + \sigma(z_t)dB_t$$

$g(z_t, t)$ – conditional distribution of z_t:

$$\frac{\partial g(z_t, t)}{\partial t} = -\frac{\partial[\mu(z_t)g(z_t, t)]}{\partial z_t} + \frac{1}{2}\frac{\partial^2[\sigma(z_t)^2 g(z_t, t)]}{\partial^2 z_t}$$

stationary distribution:

$$0 = -\frac{\partial[\mu(z)g(z)]}{\partial z} + \frac{1}{2}\frac{\partial^2[\sigma(z)^2 g(z)]}{\partial^2 z}$$

In this model:

$$0 = -\frac{\partial[s_j(a)g_j(a)]}{\partial a} + \sum_{m=1, m\neq j}^{11} (\lambda_{m,j} \cdot g_m(a) - \lambda_{j,m} \cdot g_j(a))$$

Challenges in the model

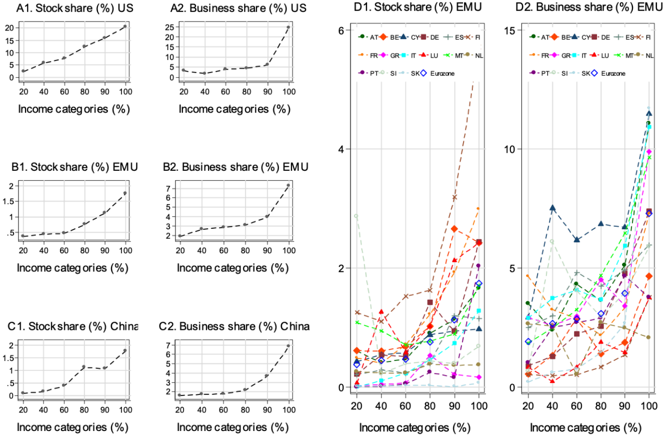

Figure: Portfolio choice and risk taking behavior

Source: Hubar, Koulovatianos, Li (2020)

The model with a portfolio choice (introduction of the risky asset)

$$\max_{(c_t, a_{t+1}, \phi_t)_{t\geq 0}} E_0 \left[\int_0^{\infty} e^{-\rho t} u(c_t) dt \right]$$

$$\tilde{R}_t dt = R_t dt + \sigma dz_t$$

New budget constraint:

$$da_t = [(1-\tau)[\phi_t \tilde{R}_t + (1-\phi_t) r_t] a_t + (1-\tau) y_{jt} - c_t + Tr] dt =$$

$$= [(1-\tau)[\phi_t R_t + (1-\phi_t) r_t] a_t + (1-\tau) y_{jt} - c_t + Tr] dt + \phi_t a_t \sigma (1-\tau) dz_t$$

Model with a jump shock hitting the whole economy

$$da_t = [(1-\tau)ra_t + (1-\tau)y_{jt} - c_t + Tr]dt - b \cdot a_t dq_t$$

where dq_t is a jump shock: $dq_t = \begin{cases} 1, & w.p. \quad \lambda dt \\ 0, & w.p. \quad 1-\lambda dt \end{cases}, \quad E_t[dq_t] = \lambda dt$

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HJB equation in this model is:

$$\rho V_j(a) = \max_{c,a}[u(c) + V_j'(a)((1-\tau)ra + (1-\tau)y_j - c + Tr))] +$$
$$+ \sum_{m=1}^{11} \lambda_{j,m}[V_m(a) - V_j(a)] + \lambda[V_j((1-b)a) - V_j(a)]$$

Kolmogorov forward equation:

$$0 = -\frac{\partial[s_j(a)g_j(a)]}{\partial a} + \sum_{m=1,m\neq j}^{11} (\lambda_{m,j} \cdot g_m(a) - \lambda_{j,m} \cdot g_j(a)) + \lambda E[g_j((1+b)a) - g_j(a)]$$

Algorithm to solve this model

1. Take a guess for the interest rate $= \frac{r_{min}+r_{max}}{2}$, where $r_{max} = \frac{\rho}{1-\tau}$, $r_{min} \to 0$.
2. Compute K, w, $y_j = z_j \times w$.
3. Pick a guess for the value function: V_{i0} for $i = 1, ..., I$.
4. Compute $(v^n)'(a_i)$, $c_i^n = (u')^{-1}[(v^n)'(a_i)]$ using first differences approximation.
5. Find v^{n+1}, if $||v^{n+1} - v^n|| < \varepsilon$, we stop.
6. Compute $g(a)$ from the KF equation
7. Ensure that asset market clears under the given interest rate. If $S > (K + B)$, we discard the upper bound, set $r_{max} = r$, update $r = (r_{min} + r_{max})/2$. If there is excess supply, we take $r_{min} = r$, update $r = (r_{min} + r_{max})/2$. We use a bisection method until $|r^{n+1} - r^n| < \varepsilon$.
8. Compute the welfare $= sum(a' * V * \Delta a)$

NMAR process for the Markov transition matrix discretization

$$log(z_t) = x_t = \rho x_{t-1} + \eta_t$$

$$\text{where} \quad \eta_t \sim \begin{cases} N(\mu_1, \sigma_1^2) & w.p. \ p_1 \\ N(\mu_2, \sigma_2^2) & w.p. \ p_2 \end{cases}$$

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NMAR:

- provides a flexible way to model shock distributions with non-Gaussian properties; can approximate almost any distribution

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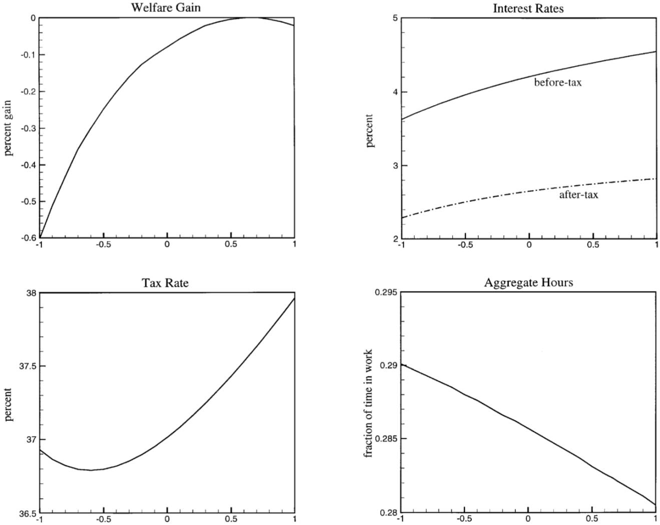
Table: Parameters of the process from Guvenen et al. (2015), Civale et al. (2017)

μ_1	μ_2	σ_1^2	σ_2^2	p_1	p_2	ρ	$E(x_t)$	$Var(x_t)$	$S(x_t)$	$K(x_t)$
0.0336	-0.3021	0.0574	1.6749	0.9	0.1	0.99	0	11.5	-0.12	3.15

Results in the Aiyagari and McGrattan (1998) model



Results (11 states): main model

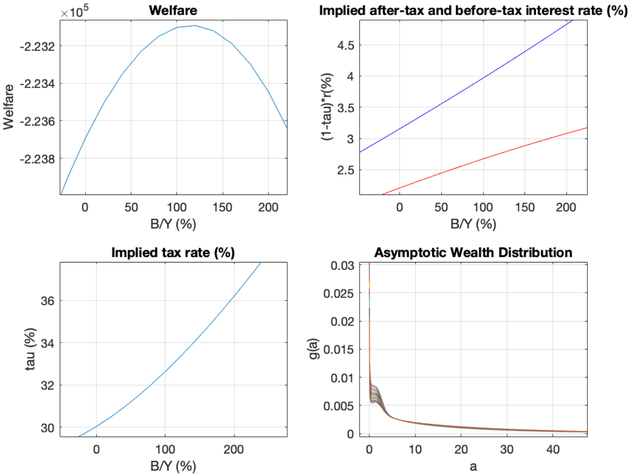


Figure: Results (11 states), $\frac{B}{Y} = 120\%$, when $a_{min} = 0; -0.3$

Effect of government debt on wealth distribution

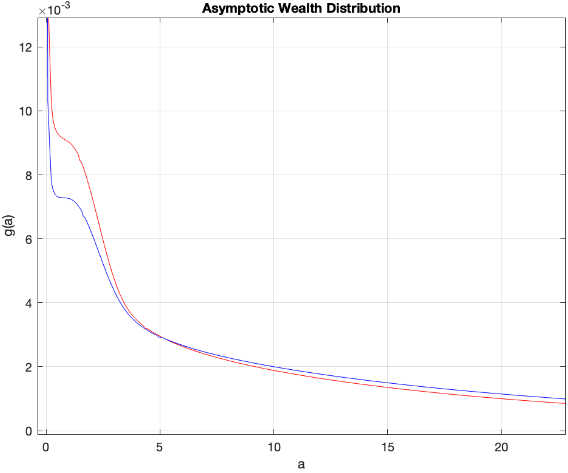


Figure: Red: debt to GDP level equal to 0%, blue: 150%

Model with a jump shock

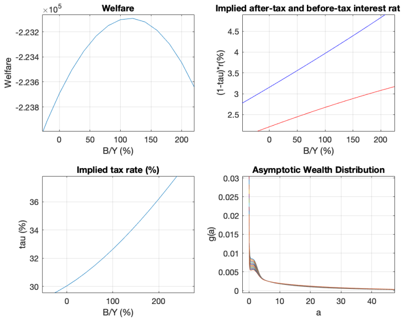


Figure: Results (11 states), no jump shock, $\frac{B}{Y} = 120\%$

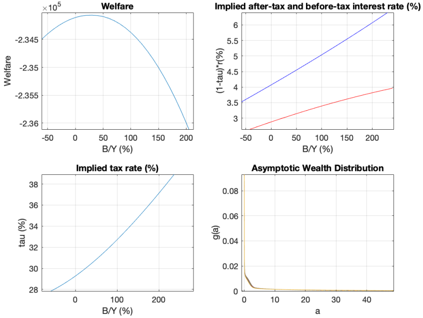


Figure: Results (11 states), jump shock, $\lambda = 0.035$, $b = 0.2328$, $\frac{B}{Y} = 20\%$

Model with a portfolio choice

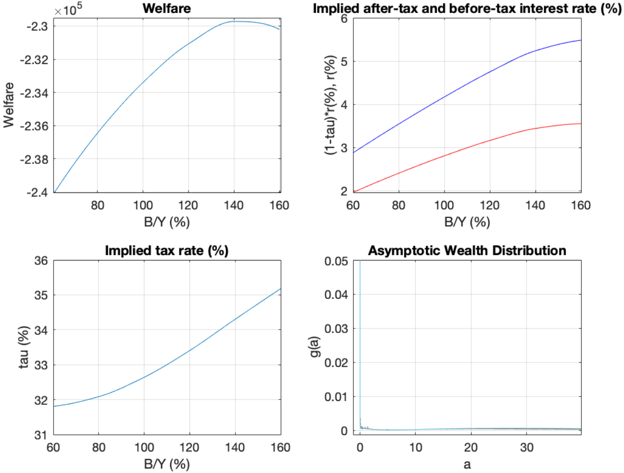


Figure: Results (11 states), portfolio choice, $\frac{B}{Y} = 140\%$

Model with a portfolio choice: Pareto tail

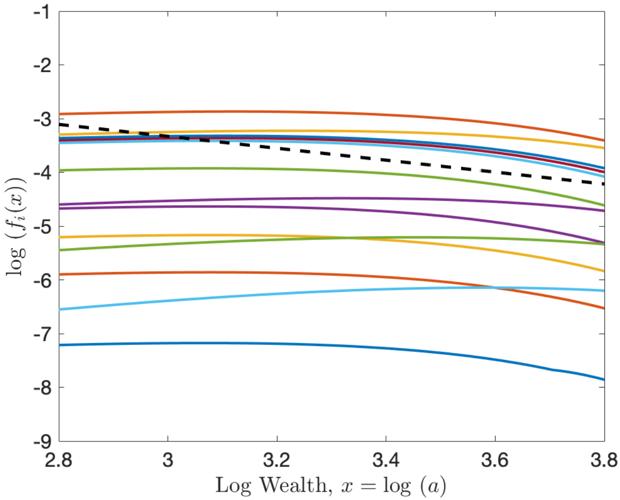


Figure: Results, portfolio choice, Pareto tail

Average return and share of the risky assets by income groups

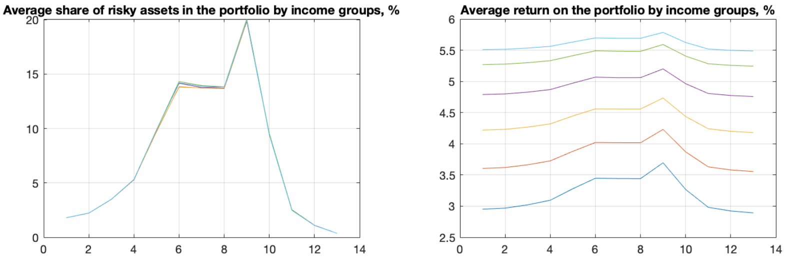


Figure: Average return and share of the risky assets by income groups

Lines represent the leverage ratio and returns under different levels of government debt

Conclusion

- I computed the heterogeneous agents model in continuous time, introduced the portfolio choice and the jump shock
- I used the process of the logarithm of endowments with non-normal innovations $\Rightarrow$ theoretical distribution meets the data
- The optimal level of the U.S. government debt varies from 120% to 140% depending on the model specification.

Further steps:

- transition dynamics, growth

Thank You for your attention!

Appendix

HJB equation in continuous time [1]

Discrete time Bellman equation:

$$V(a_t) = \max_{c_t \geq 0} \left\{ u(c_t) + \frac{1}{1+\rho} E_t[V(a_{t+1})] \right\}$$

Taking the Δt subdivision of time

$$V(a_t) = \max_{c_t \geq 0} \left\{ u(c_t)\Delta t + \frac{1}{1+\rho\Delta t} E_t[V(a_{t+\Delta t})] \right\}$$

Multiplying both sides by $1 + \rho\Delta t$

$$V(a_t) + \rho V(a_t)\Delta t = \max_{c_t \geq 0} \left\{ u(c_t)\Delta t + u(c_t)\rho(\Delta t)^2 + E_t[V(a_{t+\Delta t})] \right\}$$

$$\rho V(a_t)\Delta t = \max_{c_t \geq 0} \left\{ u(c_t)\Delta t + u(c_t)\rho(\Delta t)^2 + E_t[V(a_{t+\Delta t}) - V(a_t)] \right\}$$

Back to HJB of the model.

HJB equation in continuous time [2]

Taking the limit $\Delta t \to 0$,

$\lim_{\Delta t \to 0} (\Delta t)^{\alpha} = 0$ if $\alpha > 1$ (Asymptotic order)

$$\Rightarrow \rho V(a_t)dt = \max_{c_t \geq 0} \{u(c_t)dt + E_t[dV_t]\}$$

$$dV_t = V'(a_t)da_t = V'(a_t)[(1-\tau)ra + (1-\tau)w - c + Tr]dt$$

divide both sides by dt

$\Rightarrow$ Continuous time HJB equation:

$$\rho V(a) = \max_{c \geq 0} \{u(c) + V'(a)[(1-\tau)ra + (1-\tau)w - c + Tr]\}$$

Back to HJB of the model.

Calibration [1]

Table: Calibration for the steady state of the model

Parameter	Value	Source
γ	2	Standard
ρ	5%	Standard
δ	5%	Standard
α	$1/3$	Standard
TFP	1.1	Calibrated
r_{min}	0	Standard
r_{max}	$\rho/(1-\tau)$	Under complete markets
R	7%	Hubar et al. (2020)
		Gomes, Michaelides (2003)
b	23.28%	Koulovatianos et al. (2018)
λ	3.5%	Koulovatianos et al. (2018)

Calibration [2]

Table: Calibration for the steady state of the model

Parameter	Value/formula	Source
G^{SS}/Y^{SS}	21.7%	Aiyagari McGrattan (1998)
	16.3%	Vogel (2014)
Tr^{SS}/Y^{SS}	8.2%	Aiyagari McGrattan (1998)
	7.6%	Vogel (2014)
K^{ss}	$\left(\frac{\alpha \cdot TFP}{\delta + r^*}\right)^{1/(1-\alpha)} L$	Calibrated
τ_K	36%	Trabandt, Uhlig (2011)
τ	$\frac{\frac{G^{SS}}{Y^{SS}} + \frac{Tr^{SS}}{Y^{SS}} + r^{SS}\frac{B^{SS}}{Y^{SS}}}{1 - \delta\frac{K^{SS}}{Y^{SS}} + r^{SS}\frac{B^{SS}}{Y^{SS}}}$	Calibrated
$Gini_w$	0.78 / 85.2	SCF / Global wealth report (Credit Suisse)
ζ	$1.1 = \frac{\frac{1}{Gini}+1}{2}$	To match the Wealth Gini

Model with a jump shock: different parameters

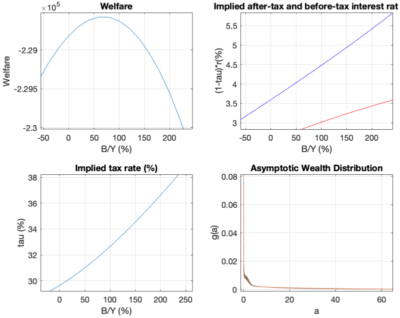


Figure: Results (11 states), jump shock, $\lambda = 0.017$, $b = 0.2328$, $\frac{B}{Y} = 60\%$

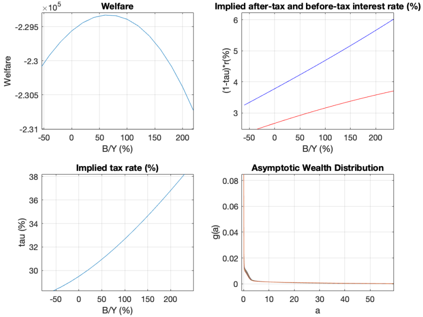


Figure: Results (11 states), jump shock, $\lambda = 0.035$, $b = 0.15$, $\frac{B}{Y} = 70\%$

HJB equation in the model with risky capital

$$\rho V_j(a) = \max_{c \geq 0} \{u(c) + V'(a)([(1-\tau)[\phi R + (1-\phi)r]a + (1-\tau)y_j$$

$$-c + Tr] + \frac{1}{2}V''(a)[\phi a \sigma (1-\tau)]^2 + \sum_{m=1}^{11} \lambda_{j,m}[V_m(a) - V_j(a)] \Big\}$$

- Optimal consumption and portfolio choices

$$c^{-\gamma} = V'(a) \quad \Rightarrow \quad c = [V'(a)]^{-1/\gamma} \quad \Rightarrow \quad u(c) = \frac{[V'(a)]^{1-1/\gamma}}{1-\gamma}$$

$$\phi = -\frac{V'_a}{V''_{aa} a} \ \frac{R - r}{\sigma^2(1-\tau)}$$

State constraint boundary conditions:

$$V'_j(\underline{a}) \geq u'((1-\tau)[\phi_j R + (1-\phi_j)r]a + (1-\tau)y_j + Tr)$$

$$V''_j(a_{max}) = \frac{-\gamma V'_j(a_{max})}{a_{max}}$$

Kolmogorov forward equation

$$0 = -\frac{\partial[s_j(a)g_j(a)]}{\partial a} + \frac{1}{2}\frac{\partial^2(g_j(a)[\sigma\phi_j a(1-\tau)]^2)}{\partial a^2} + \\ + \sum_{m=1,m\neq j}^{11} (\lambda_{m,j} \cdot g_m(a) - \lambda_{j,m} \cdot g_j(a))$$

The corresponding saving function is:

$$s_j = (1-\tau)[\phi R + (1-\phi)r]a + (1-\tau)y_j - c + Tr$$

Pareto tail

$$Pr(wealth \geq a) = 1 - G(a) = \left(\frac{m}{a}\right)^{\zeta}$$

PDF: $g(a) = \zeta m^{\zeta} a^{-\zeta-1} = \kappa a^{-\zeta-1}$.

Gini coefficient: $Gini = \frac{1}{2\zeta-1}$

Logarithm of the mass (density) of the upper tail is linear in $log(a)$:
if we have $g(a) \sim \kappa a^{-\zeta-1}$, defining $x = log(a) = h(a)$, $a = e^{x} = h^{-1}(x)$, the pdf of x is equal to $f(x) = (h^{-1}(x))' g(h^{-1}(x)) = \kappa e^{-x\zeta}$ (change of variables for distributions).
So $log(f(x)) = log(\kappa) - \zeta x$.

Pareto tail parameter of the wealth distribution is:

$$\zeta = \gamma \left(\frac{2\sigma^2(\rho - r(1-\tau))}{(R-r)^2} - 1 \right).$$

Numerical approach to solve the model

For constant $c_i = (1-\tau)ra_i + (1-\tau)[y_1 \ y_2 \ \cdots \ y_{11}] + Tr$, length of a_{grid} is I; the guess for the Value function is:

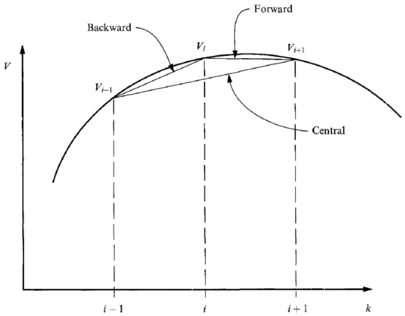
$$V\left(\begin{bmatrix} a_{min} \\ \vdots \\ a_{max} \end{bmatrix}\right) = \int_0^\infty u((1-\tau)r\begin{bmatrix} a_{min} \\ \vdots \\ a_{max} \end{bmatrix} + (1-\tau)[y_1 \ y_2 \cdots y_{11}] + Tr)e^{-\rho t}dt$$

$$V(a_{grid}) = \frac{[(1-\tau)ra_{grid} + (1-\tau)[y_1 \ y_2 \ \cdots \ y_{11}] + Tr]^{1-\gamma}}{\rho(1-\gamma)}$$

Approximated Value functions

$$V(a) = \begin{bmatrix} V_1(a_1) & V_2(a_1) & \cdots & V_{11}(a_1) \\ V_1(a_2) & V_2(a_2) & \cdots & V_{11}(a_2) \\ \vdots & \vdots & \ddots & \vdots \\ V_1(a_I) & V_2(a_I) & \cdots & V_{11}(a_I) \end{bmatrix}$$

Finite differences method



Source: Candler (1999)

Forward derivative:

$$V'_{i,j,F} = \frac{V_{i+1} - V_i}{\Delta a},$$

$$V'_{F,[1:I-1,:]} = \frac{V_{[2:I,:]} - V_{[1:I-1,:]}}{\Delta a}.$$

Backward derivative:

$$V'_{i,j,B} = \frac{V_i - V_{i-1}}{\Delta a},$$

$$V'_{B,[2:I,:]} = \frac{V_{[2:I,:]} - V_{[1:I-1,:]}}{\Delta a}.$$

Second derivative:

$$V''_{i,j} = \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta a)^2}$$

Finite differences method in the model

Due to concavity of the value function:
$V'_F < V'_B \Rightarrow (u')^{-1}(V'_F) > (u')^{-1}(V'_B) \Rightarrow C_F > C_B \Rightarrow S_F < S_B.$

$$s_{i,F} = (1-\tau)ra_i + (1-\tau)y_j - [V'_{i,F}(a)]^{-1/\gamma} + Tr \quad \text{for} \quad i = 1, ...I$$
$$s_{i,B} = (1-\tau)ra_i + (1-\tau)y_j - [V'_{i,B}(a)]^{-1/\gamma} + Tr$$

if $S_F > 0 \Rightarrow S_B > S_F > 0 \Rightarrow a \uparrow$

if $S_B < 0 \Rightarrow S_B > S_F \Rightarrow a \downarrow$

Upwind scheme: $\Rightarrow V'(a) = \mathbb{I}_{\{S_F>0\}} V'_F + \mathbb{I}_{\{S_B<0\}} V'_B + \mathbb{I}_{\{S_B \geq 0 \geq S_F\}} V'$

Finite differences method in the model [2]

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} = -\rho V_{i,j}^{n+1} + u^n + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta a}(S_{F_{i,j}}^{n})^{+} + \\ + \frac{V_{i,j}^{n+1} - V_{i-1,j}^{n+1}}{\Delta a}(S_{B_{i,j}}^{n})^{-} + \sum_{m=1, m\neq j}^{11} \lambda_{j,m}[V_{i,m}^{n+1} - V_{i,j}^{n+1}]$$

$$\textit{where } \Delta = \frac{\Delta a}{max[(1-\tau)(ra_{grid} + [y_1 \;\, y_2 \;\, \cdots \;\, y_{11}]) + Tr]} \;\; \text{(CFL condition)}.$$

Matrix representation of HJB equation:

$$\frac{V^{n+1} - V^{n}}{\Delta} = -\rho V^{n+1} + u^n + (A_1 + A_2)V^{n+1}$$

Finite differences method in the model [3]

In the model with portfolio choice:

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta} = -\rho V_{i,j}^{n+1} + u^n + \frac{V_{i+1,j}^{n+1}-V_{i,j}^{n+1}}{\Delta a}(S_{F_{i,j}}^{n})^{+} + \frac{V_{i,j}^{n+1}-V_{i-1,j}^{n+1}}{\Delta a}(S_{B_{i,j}}^{n})^{-}$$

$$+\frac{(\sigma(1-\tau))^2}{2}(\phi_{i,j}a_i)^2\frac{V_{i+1,j}^{n+1}-2V_{i,j}^{n+1}+V_{i-1,j}^{n+1}}{(\Delta a)^2} + \sum_{m=1,m\neq j}^{11}\lambda_{j,m}[V_{i,m}^{n+1}-V_{i,j}^{n+1}]$$

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta} = -\rho V_{i,j}^{n+1} + u^n + V_{i-1,j}^{n+1}x_{i,j} + V_{i,j}^{n+1}y_{i,j} +$$

$$+V_{i+1,j}^{n+1}z_{i,j} + \sum_{m=1,m\neq j}^{11}\lambda_{j,m}V_{i,m}^{n+1}$$

$$x_{i,j} = \frac{(S_{B_{i,j}}^{n})^{-}}{\Delta a} + \frac{(\sigma(1-\tau))^2}{2}\frac{(\phi_{i,j}a_i)^2}{(\Delta a)^2},$$

$$y_{i,j} = \frac{(S_{F_{i,j}}^{n})^{+}+(S_{B_{i,j}}^{n})^{-}}{\Delta a} - (\sigma(1-\tau))^2\frac{(\phi_{i,j}a_i)^2}{(\Delta a)^2} - \sum_{m=1,m\neq j}^{11}\lambda_{j,m},$$

$$z_{i,j} = \frac{(S_{F_{i,j}}^{n})^{+}}{\Delta a} + \frac{(\sigma(1-\tau))^2}{2}\frac{(\phi_{i,j}a_i)^2}{(\Delta a)^2}$$

Finite differences method in the model [4]

In the model with a jump shock:

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} = -\rho V_{i,j}^{n+1} + u^n + \lambda[V_j^n((1-b)a) - V_j^n] + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta a}(S_{F_{i,j}}^n)^+ + \frac{V_{i,j}^{n+1} - V_{i-1,j}^{n+1}}{\Delta a}(S_{B_{i,j}}^n)^- + \sum_{m=1, m\neq j}^{11} \lambda_{j,m}[V_{i,m}^{n+1} - V_{i,j}^{n+1}];$$

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} = -\rho V_{i,j}^{n+1} + u^n + \lambda[V_j^n((1-b)a) - V_j^n] + V_{i-1,j}^{n+1} x_{i,j} + + V_{i,j}^{n+1} y_{i,j} + V_{i+1,j}^{n+1} z_{i,j} + \sum_{m=1, m\neq j}^{11} \lambda_{j,m} V_{i,m}^{n+1}$$

$x_{i,j} = \frac{(S_{B_{i,j}}^n)^-}{\Delta a}$,
$y_{i,j} = \frac{(S_{F_{i,j}}^n)^+ + (S_{B_{i,j}}^n)^-}{\Delta a} - \sum_{m=1, m\neq j}^{11} \lambda_{j,m}$,
$z_{i,j} = \frac{(S_{F_{i,j}}^n)^+}{\Delta a}$

Finite differences method in the model [5]

$$\frac{V^{n+1} - V^n}{\Delta} = -\rho V^{n+1} + u^n + (A_1 + A_2) V^{n+1}$$

where

$$V^{n+1} = \begin{bmatrix} V^{n+1}_{1,1} \\ \vdots \\ V^{n+1}_{I,1} \\ V^{n+1}_{1,2} \\ \vdots \\ V^{n+1}_{I,2} \\ \vdots \\ V^{n+1}_{1,11} \\ \vdots \\ V^{n+1}_{I,11} \end{bmatrix}, \quad u^n = \begin{bmatrix} u(c^n_{1,1}) \\ \vdots \\ u(c^n_{I,1}) \\ u(c^n_{1,2}) \\ \vdots \\ u(c^n_{I,2}) \\ \vdots \\ u(c^n_{1,11}) \\ \vdots \\ u(c^n_{I,11}) \end{bmatrix}$$

Matrix A_1

$$
\begin{bmatrix}
\begin{bmatrix}
y_{1,1} & z_{1,2} & 0 & \cdots & 0 \\
x_{2,1} & y_{2,1} & z_{2,1} & \cdots & 0 \\
0 & x_{3,1} & y_{3,1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & y_{I,1}
\end{bmatrix} & & 0 \\
& \ddots & \\
0 & & \begin{bmatrix}
y_{1,11} & z_{1,11} & 0 & \cdots & 0 \\
x_{2,11} & y_{2,11} & z_{2,11} & \cdots & 0 \\
0 & x_{3,11} & y_{3,11} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & y_{I,11}
\end{bmatrix}
\end{bmatrix}
$$

Matrix A_2

$$\begin{bmatrix} \begin{bmatrix}\lambda_{1,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{1,1}\end{bmatrix} & \begin{bmatrix}\lambda_{1,2} & & 0 \\ & \ddots & \\ 0 & & \lambda_{1,2}\end{bmatrix} & \cdots & \begin{bmatrix}\lambda_{1,11} & & 0 \\ & \ddots & \\ 0 & & \lambda_{1,11}\end{bmatrix} \\ \begin{bmatrix}\lambda_{2,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2,1}\end{bmatrix} & \begin{bmatrix}\lambda_{2,2} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2,2}\end{bmatrix} & & \\ \vdots & & \ddots & \\ \begin{bmatrix}\lambda_{11,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{11,1}\end{bmatrix} & \cdots & & \begin{bmatrix}\lambda_{11,11} & & 0 \\ & \ddots & \\ 0 & & \lambda_{11,11}\end{bmatrix} \end{bmatrix}$$

Kolmogorov forward equation and finite differences approximation

In the model with a portfolio choice:

$$0 = -\frac{(S^n_{F_{i,j}})^+ g_{i,j} - (S^n_{F_{i-1,j}})^+ g_{i-1,j}}{\Delta a} - \frac{(S^n_{B_{i+1,j}})^- g_{i+1,j} - (S^n_{B_{i,j}})^- g_{i,j}}{\Delta a} +$$

$$+\frac{(\sigma(1-\tau))^2}{2}(\phi_{i,j}a_i)^2\frac{g_{i+1,j} - 2g_{i,j} + g_{i-1,j}}{(\Delta a)^2} + \sum_{m=1,m\neq j}^{11}(\lambda_{m,j}\cdot g_{i,m} - \lambda_{j,m}\cdot g_{i,j})$$

where saving policy function from the HJB equation
$S_j(a) = (1-\tau)[\phi_j R + (1-\phi_j)r]a + (1-\tau)y_j - c_j(a) + Tr.$

Stationary wealth distribution

Solution for the density function:

$$A'g(a) = 0$$

$$g(a) = \left[g_{1,1} \cdots g_{I,1} g_{1,2} \cdots g_{I,2} \cdots g_{1,11} \cdots g_{I,11}\right]'$$

$$A' \cdot g(a) = \begin{bmatrix} y_{1,1} + \lambda_{1,1} + \varepsilon & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} g_{1,1} \\ \vdots \\ \vdots \\ g_{I,11} \end{bmatrix} = \begin{bmatrix} 0 + \varepsilon \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

The solution to this system is $g(a) = (A')^{-1} 0_{I*J \times 1}$, then this vector is normalized so that the sum is equal to 1.

Calibration of the labor endowment (income) process

Stochastic process of the logarithm of the labor endowment:

$$log(z_t) = \rho log(z_{t-1}) + \sigma_e \varepsilon_t,$$

where $\varepsilon_t \sim N(0,1)$, $\sigma_e = \sigma\sqrt{1-\rho^2}$;
Asymptotic distribution of $x = log(z)$ (if $|\rho| < 1$) is the following:

$$x_\infty \sim N\left(0, \frac{\sigma^2}{1-\rho^2}\right)$$

Markov transition matrix is given by

$$M = \begin{pmatrix} \pi_{1,1} & \cdots & \pi_{1,11} \\ \vdots & \ddots & \vdots \\ \pi_{11,1} & \cdots & \pi_{11,11} \end{pmatrix}$$

where $\pi_{ij} = Pr(X_{t+1} = x_j | X_t = x_i)$

Discretization of the Markov transition matrix

Choose $x = \{x_1, ...x_{11}\}$,
$x_1 = -\Omega\frac{\sigma}{\sqrt{1-\rho^2}}$; $x_{11} = \Omega\frac{\sigma}{\sqrt{1-\rho^2}}$, where Ω is a number of standard deviations to cover the support

distance is equal to $d = \frac{x_{11}-x_1}{n-1}$, $\varepsilon_{t+1} = \frac{x_{t+1}-\rho x_t}{\sigma}$

$$\pi_{i,j} = Pr(x_{t+1} = x_j | x_t = x_i) =$$
$$\begin{cases} \Phi\left(\frac{x_j+\frac{d}{2}-\rho x_i}{\sigma}\right) - \Phi\left(\frac{x_j-\frac{d}{2}-\rho x_i}{\sigma}\right), & j = \{2, ...10\} \\ \Phi\left(\frac{x_j+\frac{d}{2}-\rho x_i}{\sigma}\right), & j = 1 \\ 1 - \Phi\left(\frac{x_j+\frac{d}{2}-\rho x_i}{\sigma}\right), & j = 11 \end{cases}$$

where Φ is CDF of the standard normal.

Markov transition matrix discretization

$$\pi_{i,j} = Pr(x_{t+1} = x_j | x_t = x_i) =$$

$$\begin{cases} p_1 * \left[\Phi\left(\frac{x_j + \frac{d_j}{2} - \rho x_i - \mu_1}{\sigma_1} \right) - \Phi\left(\frac{x_j - \frac{d_{j-1}}{2} - \rho x_i - \mu_1}{\sigma_1} \right) \right] + \\ + p_2 * \left[\Phi\left(\frac{x_j + \frac{d_j}{2} - \rho x_i - \mu_2}{\sigma_2} \right) - \Phi\left(\frac{x_j - \frac{d_{j-1}}{2} - \rho x_i - \mu_2}{\sigma_2} \right) \right] \\ p_1 * \Phi\left(\frac{x_j + \frac{d_j}{2} - \rho x_i - \mu_1}{\sigma_1} \right) + p_2 * \Phi\left(\frac{x_j + \frac{d_j}{2} - \rho x_i - \mu_2}{\sigma_2} \right), & \text{for } j = 1 \\ 1 - p_1 * \Phi\left(\frac{x_j + \frac{d_{j-1}}{2} - \rho x_i - \mu_1}{\sigma_1} \right) - p_2 * \Phi\left(\frac{x_j + \frac{d_{j-1}}{2} - \rho x_i - \mu_2}{\sigma_2} \right), & \text{for } j = 11 \end{cases}$$

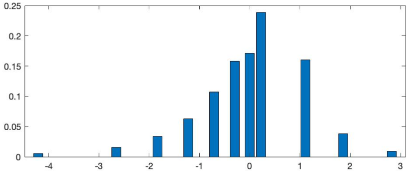


Figure: Logarithm of labor endowments under Extended Tauchen method

Markov transition matrix discretization

The algorithm

1. Choose the arbitrary number of states (I choose 11)
2. Make a guess on the grid of x
3. Compute the Markov transition matrix
4. Evaluate the distance between the targets and the moments of the Markov chain:
   $|m(\theta) - \hat{m}(x, P)| = (m(\theta) - \hat{m}(x, P))' W (m(\theta) - \hat{m}(x, P))$, where W is a weighting matrix so that the distance is the sum of squared percentage deviations of each moment from its target.
5. Iterate the procedure to minimize the distance.
   At each iteration we map the discrete Markov process into the relevant set of moments from the table.

NMAR: moments of x

$$E(x) = \sum_{i=1}^{n} \Pi_i z_i$$

$$S(x) = \frac{\sum_{i=1}^{n} \Pi_i (z_i - E(x))^3}{Std(x)^3}$$

$$Var(x) = \sum_{i=1}^{n} \Pi_i (z_i - E(x))^2$$

$$K(x) = \frac{\sum_{i=1}^{n} \Pi_i (z_i - E(x))^4}{Var(x)^2}$$

$$\rho(x) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \Pi_i T_{ij} (z_i - E(x))(z_j - E(x))}{Var(x)}$$

NMAR: moments of e

$$E_{i,j} = z_j - \rho(x) z_i$$

$$E(e) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Pi_i T_{ij} E_{i,j}$$

$$S(e) = \frac{\sum_{i=1}^{n} \Pi_i T_{i,j} (E_{i,j} - E(e))^3}{Std(e)^3}$$

$$Var(e) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Pi_i T_{i,j} (E_{i,j} - E(e))^2$$

$$K(e) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \Pi_i T_{i,j} (E_{i,j} - E(e))^4}{Var(e)^2}$$